# Mathematics Tutor Workshop

Resources and Materials

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CAL MARITIME

# Summary

This material has been created and compiled for Mathematics Tutor training workshops to supplement tutor training at the CSU Maritime Academy. The resources herein are designed to be used in a workshop setting, with activities led by mathematics faculty.

# Contents

# CHAPTER 1

# What are Teaching, Learning, and Pedagogy?

Before we get into tutoring specifically, we need to cover some basic vocabulary and concepts relevant to the learning process. This is important to ensure we all can start with the same foundation.

- **Teaching** is the practice or art of imparting information from an instructor to a student. We like to think of this as an act of *empowerment* that expands intellectual and physical capabilities for students. Teaching refers to the instructor's role in the learning process. Teaching is not passive on the part of the instructor, and must be adapted to students and educational settings (classrooms, number of students/tables/desks/whiteboards/etc.)
- **Learning** is what happens on the student side: it is the expansion of a student's knowledge or skill base. It is not a passive process on the part of the student. Students must engage with the material in order to learn.
- **Pedagogy** refers to the practices of teaching. While the classic university pedagogical structure has historically been a lecture-style teaching method, many instructors adopt strategies that involve *active learning* within their classroom. *Active learning* refers to activities in which students must engage and participate in the process of discovery. This can be done with worksheets, board work, group work, and a number of other tools.
- **Assessment** refers to the measurement of student comprehension of a topic or learning objective using assignments or tasks.
- **Learning objectives/outcomes** refer to the content students are expected to learn in a course.

As a tutor in this workshop, our goals for you are:

1. To provide some contextual framework for how teaching and learning happen (and don't!);
2. To assess your strengths and weaknesses as an instructor;
3. To provide you with some pedagogical tools to improve your effectiveness as a tutor;
4. To provide you with an inventory for our mathematics courses that can help you assess your own knowledge base and find information; and
5. To have fun! Teaching is fun, when done right. Learning can (and ideally, should) be too.

## Learning

On the next pages, you'll find a summary of what is known as Bloom's Taxonomy, which is a classification system for how educators think of measuring student understanding of a concept. [1] One important aspect to note is that when teachers create assignments and tests, they often have this taxonomy in mind and are attempting to evaluate student performance on these specific levels. Instructors often aim to assess ALL levels of the taxonomy during a course as well as all student learning outcomes for the particular course. Notice the labeling of these learning levels in Bloom's Taxonomy: some are consider "high school level" and others "college level." At universities, classes are often not geared towards simple memory exercises and repeating of ideas but instead towards more detailed *analyses* and *expansion* or *creation* of new ideas.

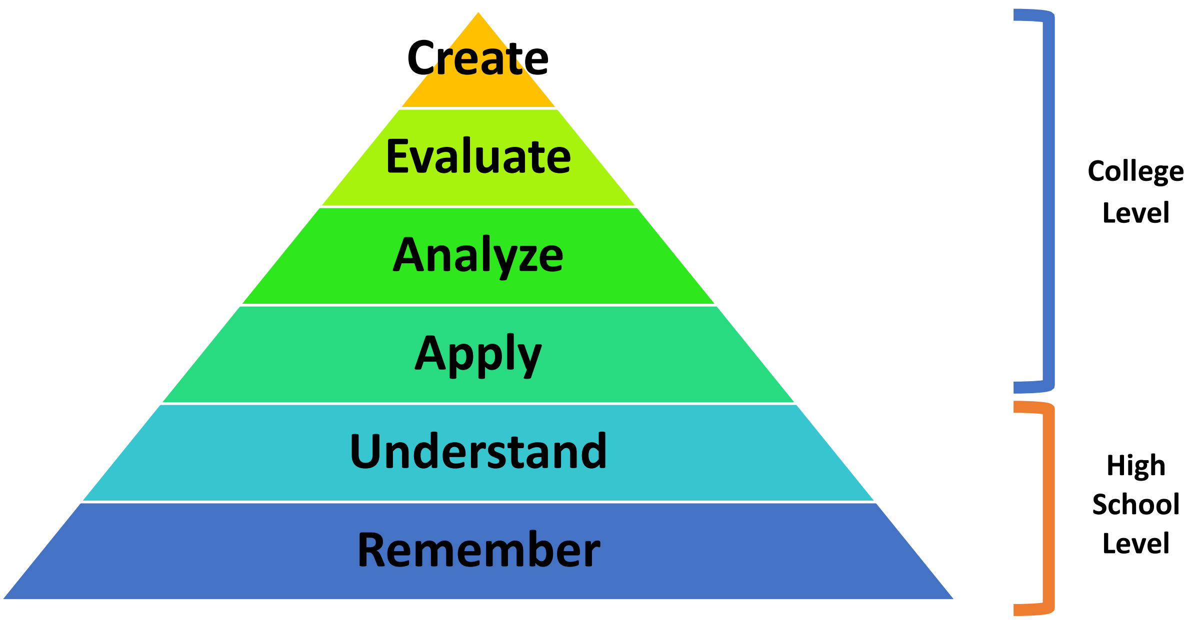
[1] This handout comes from The Academic Success Center, Texas A&M University and the subsequent worksheet was adapted from their accompnaying worksheet

# Bloom’s Taxonomy

Academic Success Center
UNDERGRADUATE STUDIES

## Levels of Understanding

Bloom’s Taxonomy is a hierarchy of learning levels. It is best represented as a pyramid where the foundation of learning is shown at the bottom, with increasingly higher-order learning as you climb toward the top. The purpose of Bloom’s Taxonomy is to make a distinction between different levels of learning so that students can study at appropriate learning levels. In high school, you might have been responsible for acquiring basic knowledge, while college courses require more application, analysis, and evaluation, and possibly even the creation of something new with what you are learning.




Most students come into college with strong study skills in how to remember and understand concepts. The challenge is to create new active study strategies that enable students to take what they know and process it at a higher level. It is important to study at the level at which you will be expected to demonstrate competency.

| | |
|---|---|
| **Create** | **Combining parts to make a new whole**<br>*Build, combine, formulate, devise, change, adapt, construct, produce* |
| **Evaluate** | **Judging the value of information or ideas**<br>*Validate, justify, critique, rate, prioritize, select, assess, monitor* |
| **Analyze** | **Breaking down information into basic parts**<br>*Classify, divide, differentiate, research, discover, simplify, dissect* |
| **Apply** | **Applying the facts, rules, concepts, and ideas**<br>*Practice, implement, develop, solve, generalize, operate, plan* |
| **Understand** | **Understanding what the facts mean**<br>*Discuss, paraphrase, infer, interpret, outline, review, organize* |
| **Remember** | **Recognizing and recalling facts**<br>*Define, list, name, recognize, match, choose, show, find* |

Now let's do a short activity to think about how this taxonomy can be applied to a real setting. On the next page, you'll find a worksheet.

# Recognizing Learning Levels

Directions: Using your knowledge of Bloom's Taxonomy, identify which level of understanding is required to answer the following statements/questions. (Refer to the Bloom's handout for clarification.)

- Remembering
- Understanding
- Applying
- Analyzing
- Evaluating
- Creating

1. Describe the significance of the Corps of Cadets to Cal Maritime.
2. What year was Cal Maritime founded?
3. Explain the role of the Commandants.
4. Suppose Cal Maritime wants to add a new college. What college would you create and why?
5. Your friend argues that Cal Maritime should not have joined the CSU. Do you agree? Why or why not?
6. Who was the first president of Cal Maritime?
7. Cal Maritime is a university defined by six core values. Break down each, describing how they relate to one another and create a unified University.
8. You meet Alicia, a classmate, and notice she is wearing her salt & peppers. What can you now conclude about Erin?
9. Why is a horizontal line a function while a vertical line is not?
10. How do we physically interpret the derivative?

## Assessing Knowledge and Learning

While the primary person involved in assessing student knowledge and learning is the instructor of the course, your role as a tutor means that you should also be trying to informally assess learning and knowledge as you go. An effective teacher does NOT just to supply answers to students but rather to helps them learn and develop independence and confidence in a subject (instead of dependence or helplessness). One way to frame this is to ask: has an instructor taught a student if the student cannot correctly demonstrate the concept on their own? The answer is no. And the responsibility lies on both the instructor and the student.

So how do we figure out if teaching and learning is happening? You can think of this as a multi-step process that models good study habits. Good study habits involve figuring out what you know and do not know, focusing your time on studying what you don't know, then testing or assessing yourself on this knowledge (figuring out what you know and don't know, again!). A teacher is a guide in that process and one way to summarize the way you might assess knowledge and learning is:

1. **Determine the foundation:** what does the student know and where are they getting lost?
2. **Determine the progress:** after an explanation is given or task is completed, has the student absorbed this? Usually the best way to do this is to give the student a simple example to do on their own. If a student is really struggling, keep the example very simple.
3. **Solidify the understanding:** again, there are many ways to do this, but one easy way is to give them a few more examples that are different or slightly harder. Another way is to ask questions that require the student to explain.
4. **Test the understanding or internalization:** this gets at a deeper understanding. Can the student extrapolate to a more complex example or something that looks different? Can they do this without help? Can they explain in words?

As you look at this process, you might be thinking a bit about Bloom's Taxonomy. You might also be thinking that you won't have enough time to do a 4-step process in every a tutoring session. The good news is that you're not the only learning guide the student is working with and even if you only get to parts of this process, that is still helpful for a student. We'll get into how this process might unfold in a tutoring scenario and discuss effective tutoring practices in the next chapter.

# CHAPTER 2

# Tutoring Practices

So you've been hired as a tutor and we've talked a little bit about teaching vs. learning and learning taxonomy ... now what? You probably already have an idea of what tutoring looks like and how you might go about tutoring. You may have also received some general tutor training that set some of the ground rules and expectations for your position. The next phase of this is to prepare yourself for what might actually happen IRL (in real life) and how tutoring in math specifically might be different from other subjects.

## 2.1 What does tutoring look like?

Each tutor and student interaction will be different, but you'll likely experience a variety of students with a variety of questions. It's important to remember that each student comes with their own mathematical foundations and background and may think in ways that are *very* different from how you think. This can lead to real challenges in effective communication and tutoring. To illustrate this, we've written a script (see next page) for you and it's been vetted by some of your favorite faculty and staff members. Let's read through it together.

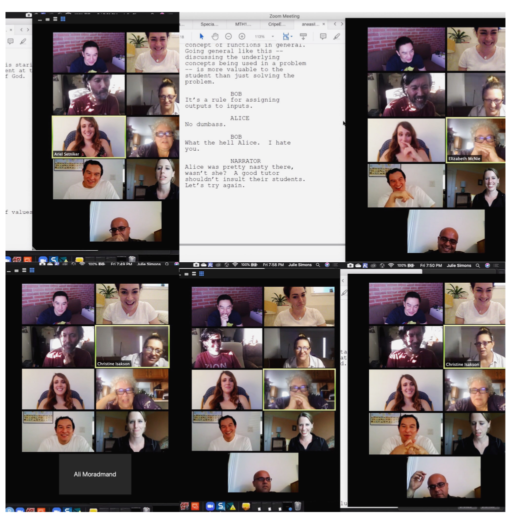



Figure 2.1: Your favorite Cal Maritime faculty and staff, starring in the debut of "An Easily Imagined Encounter Between a Tutor and Student." Possible future viral video content!

AN EASILY IMAGINED ENCOUNTER BETWEEN A TUTOR AND STUDENT

by

Taiyo Inoue

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California Maritime
Academy

FADE IN:

INT. TUTORING CENTER

Alice and Bob meet at the tutoring center. Alice is staring at her phone while Bob is just arriving. The narrator is not physically present at the tutoring center -- think of her/him as the voice of God.

BOB
Hello there Alice!

ALICE
Oh, hi Bob.

BOB
I could sure use some help with MTH 100. My professor assigned some homework and I don't understand one of the problems.

ALICE
No sweat. What's the statement of the problem you are having trouble with?

BOB
Well, I'm given a table of values. Let me write it on the board.

Bob writes on the whiteboard the following table of values:

| $x$ | $y$ |
|---|---|
| 1 | 3 |
| 2 | 3 |
| 3 | 4 |
| 3 | $-4$ |
| 4 | 3 |

Alice, feeling bored during the 15 second duration in which Bob is writing the table of values, pulls out her phone and giggles. Bob looks over at Alice and feels a little bit dejected.

BOB
The question is asking me if this table of values represents a func...tion.. or... uh, Alice, are you listening?

NARRATOR
An effective tutor should demonstrate dedication and empathy. The phone feels like a rejection. Alice should try that again:

Rewind time.

Alice sits with her boredom while Bob writes the values on the board. Alice feels frustration at how slowly Bob is writing, but tries to practice patience.

BOB
The question is asking me if this table of values represents a function or not.

ALICE
Oh, that’s easy. It’s not a function. Done.

Alice returns to her phone and Bob happily writes down the answer, but still has no idea why he’s writing what he’s writing.

NARRATOR
This encounter was a failure. An effective tutor is never a mere dispensary for homework solutions. Rather, an effective tutor gently guides their students toward the solution. Let’s try that again.

Rewind time.

BOB
The question is asking me if this table of values represents a function or not.

ALICE
Ah, I see. OK! Can you tell me what a function is?

NARRATOR
This is a good move! As a tutor, it’s better to ask questions than give answers. Also, notice how instead of talking about just the problem-at-hand, Alice has expanded the scope of the discussion to be about the concept of functions in general. Going general like this -- discussing the underlying concepts being used in a problem -- is more valuable to the student than just solving the problem.

BOB
It’s a rule for assigning outputs to inputs.

ALICE
No dumbass.

BOB
What the hell Alice. I hate you.

NARRATOR
Alice was pretty nasty there, wasn’t she? An effective tutor shouldn’t insult their students. Let’s try again.

Rewind time.

BOB
The question is asking me if this table of values represents a function or not.

ALICE
Ah, I see. OK! Can you tell me what a function is?

BOB
It’s a rule for assigning outputs to inputs.

ALICE
Very close. Close enough for now. In this table, there are two columns. What do you think those represent?

NARRATOR
Notice that though the answer Bob provided was not quite on the nose, it was close enough that progress could be made.

BOB
I guess one is the input and the other is the output?

ALICE
Oh, that’s very good! But which is which?

BOB
I don’t know.

ALICE
OK. Well, by convention, we take the column labelled $x$ to represent the input values while the column labeled $y$ represents the corresponding output values.

Bob nods his head up and down indicating that he understands what’s happening, but his face seems to indicate he’s perturbed. But Alice suspects that Bob doesn’t really understand and so follows with a question.

ALICE
Do you know what I mean when I say, "by convention?"

NARRATOR
Sometimes students will withhold their ignorance. But you can often tell when this happens through tells like body language or vocal giveaways. An effective tutor won’t let this go.

BOB
My professor says that thing... "by contention," all the time but I have no idea what it means.

ALICE
No. Not "by contention." Rather "by convention." In this case, it just means that humanity has decided rather arbitrarily to make the letter $x$ represent inputs to functions and the letter $y$ represent outputs.

BOB
Well, wait, aren't $x$ and $y$ also the names of the lines in like, the plane?

ALICE
Yes, but let's not get into that now.

NARRATOR
Why not Alice? Bob is making connections. These sorts of connections are valuable for students and should be encouraged. Let's try that again:

Rewind time.

BOB
Well, wait, aren't $x$ and $y$ also the names of the the lines in like, the plane?

ALICE
Yes! And that's relevant.

BOB
Really, why?

ALICE
Well, what does the $xy$-plane have to do with functions? One moment, let me draw the $xy$-plane.

Alice draws the familiar picture of the $xy$-coordinate plane.

BOB
Oh, the $xy$-plane is a function. Right?

ALICE
No. Absolutely not. A plane, or any other geometric object, is not a function.

NARRATOR
Maybe this sounds harsh. And maybe Alice should be expected to go a little easier on Bob. Nonetheless, it's better to squash errors like this definitively and as soon as possible so that the student can get the bad idea out of their head.

ALICE
Let me ask again: what does the $xy$-plane have to do with functions?

BOB
Well, what about like graphs?

ALICE
Oh, good, I'm glad you mentioned graphs.

BOB
Right, because graphs are functions.

ALICE
NO! That's a category error. Graphs are curves and functions are rules for associating outputs to inputs. How can a curve be a rule?

BOB
Uh, what? I don't know.

ALICE
Then why did you say graphs are functions?

BOB
Uh I don't... uh... OK.

NARRATOR
But clearly it's not OK. Alice sure is being pedantic. I mean sure -- technically graphs are not functions. But they are excellent representations of functions. And the correspondence between functions and graphs is pretty tight! This sort of slip-up is small enough that Alice shouldn't get bogged down by it. How could Alice have done better?

Rewind time.

BOB
Right, because graphs are functions.

ALICE
Well almost. Graphs are great representations of functions -- kind of in the same way that a photograph of a tree is not the same as a tree. But we can totally work with graphs to study functions in the same way we can work with the photograph to study trees.

NARRATOR
A good analogy can go a long way.

Silence settles. Alice appears to be spacing out.

BOB
Alice, you there?

ALICE
Give me a second to think...

NARRATOR
Better to stop and think rather than say something stupid and rash.

ALICE
Is every curve that I draw in the plane the graph of a function?

BOB
Uh... I suspect that because you are asking that question, that the answer is "no" but I don't really know otherwise.

ALICE
Have you learned something called the "Vertical Line Test"?

BOB
Yes!

ALICE
OK! Great. Can you tell me what you know about it?

BOB
It's like, if uh, a straight line, like hits the graph more than once, then it's not a function.

NARRATOR
Alice is in a pedantic panic now. But she maintains her cool and chooses to charitably correct Bob's mistakes.

ALICE
OK... any straight line works for the *Vertical* Line Test?

BOB
Oh. Ha. No, of course the line should be vertical.

ALICE
OK. And if every vertical line intersects a given curve just once is the curve a function.

BOB
Yep.

ALICE
Good, can you tell me why that is so?

BOB
Well. You want me to tell you why? I thought that was just a rule.

ALICE
Well, usually there are good reasons for the rules in math.

BOB
I guess...

ALICE
OK, how about this. I'm going to draw two $xy$–planes. On one I'll draw a curve which satisfies the Vertical Line Test, and on the other I'll draw a curve which does not satisfy the Vertical Line Test.

NARRATOR
Maybe Alice should let Bob try to draw these curves and get him more actively involved. Making students active participants in learning (as opposed to just listening and being passive) is a good practice. Let's try that again:

Rewind time.

ALICE
OK, how about this. I'm going to draw two $xy$-planes. On one I want you to draw a curve which satisfies the Vertical Line Test, and on the other I want you to draw a curve which does not satisfy the Vertical Line Test.

BOB
OK

Alice draws two $xy$-planes, one on the left and one on the right. Bob proceeds to draw on left plane a squiggly curve which passes the Vertical Line Test, while on the right plane he draws an $S$ shaped curve which doesn't pass the Vertical Line Test. Alice points to the left-plane:

ALICE
Now the Vertical Line Test says this one is the graph of a function and that one is not. Agreed?

BOB
Agreed.

ALICE
So why is this the graph of a function?

BOB
Because of the Vertical Line Test.

ALICE
What? No. That's circular thinking! You can't justify that the statement is true by using the truth of the statement. That would be like saying, "All squares are rectangles because all squares are rectangles," which is not good reasoning at all.

BOB
Uh, OK, sure.

ALICE
So why is this the graph of a function?

BOB
I don't know.

ALICE
OK. Let's take something even simpler because sometimes it's the simplest examples which are the most enlightening.

NARRATOR
It's true. Simple examples can isolate ideas and bring out misunderstandings of those ideas quickly.

Alice draws another coordinate plane and in the corner writes $f(x) = 2$.

ALICE
What does the graph of this function $f(x) = 2$ look like?

BOB
Oh, I know. It's a line.

ALICE
Can you be more descriptive?

BOB
It's a horizontal line.

ALICE
Yes! Can you draw the graph for me?

Bob draws the horizontal line $y = 2$.

BOB
Is that right?

ALICE
Yes, absolutely. Now suppose I erase the formula for the function...

Alice erases the $f(x) = 2$ bits.

ALICE
How could you use this graph to tell me the value of, let's say, $f(0)$?

BOB
Well, $f(0) = 2$ because it's always 2.

Alice takes a deep breath.

ALICE
Yes, but what is it about the graph that communicates that fact?

BOB
I don’t know. I’m lost.

ALICE
Ugh. OK. You need to go back and review the relationship between a function and its graph. I don’t have time to teach you things you should have learned in 11th grade. Come back to me when you’re prepared.

Alice pulls her phone out and thumbs it violently, looking at the screen but not perceiving it. Bob is shocked, frustrated and sad.

NARRATOR
Woops. Frustration has boiled over and Alice has snapped. She is trying to place blame on Bob, or Bob’s old teachers, or Bob’s old schooling. But really, this is just a way of deflecting Alice’s own responsibility as a tutor onto someone or something else. And look where we have ended up after this blowup: Alice is frustrated not just with Bob, but with herself, and Bob is totally dejected and left more confused than when he started. Let’s try that again. Persevere Alice!

Rewind time.

BOB
I don’t know. I’m lost.

ALICE
OK. In the equation $f(0) = 2$, what is the input and what is the output?

NARRATOR
Good job Alice. Back on track. Notice that in response to Bob’s "I don’t knows", Alice is digging deeper into the concept of a graph to try and find precisely where Bob’s misunderstanding or missed connection is. This is good practice!

BOB
The input is 0 and the output is 2.

ALICE
Right. Now how does the graph communicate this?

BOB
I don’t know.

ALICE
OK. You’ve seen your professor write things like "$f(x) = y$" before, yes?

BOB
Yeah. Oh, and sometimes my professor writes the $y$ inside the $f$, but with like, a $-1$ at the top.

ALICE
Oh, like this: $f^{-1}(y) = x$.

BOB
Yes! But what is that?

Alice realizes she doesn’t know. She knows that $f^{-1}$ refers to the inverse of $f$, but she doesn’t remember what it means. But she doesn’t want to look ignorant so she confidently but wrongly replies:

ALICE
Oh, uh, that just means you take the reciprocal of $f$...

NARRATOR
Alice should not speak as though she knows what she is talking about if she does not. Rather, she should admit that she doesn’t know -- there is no shame in that. Let’s try this again:

Rewind Time.

BOB
Yeah. Oh, and sometimes my professor writes the $y$ inside the $f$, but with like, a $-1$ at the top.

ALICE
Oh, like this: $f^{-1}(y) = x$.

BOB
Yes! But what is that?

ALICE

Ummm, honestly, I'm not totally sure. I'll look it up after we finish this problem. But let's not get distracted! In $f(x) = y$, what is the input and what is the output?

BOB

The input is $x$ and the output is $y$.

ALICE

Yes. So doesn't it make sense that the input is recorded on the $x$-axis and the output is recorded on the $y$-axis?

BOB

Hm, yes.

ALICE

OK. So let's look at $f(0) = 2$ again and compare it with $f(x) = y$. What's playing the role of $x$ and $y$ in this equation?

BOB

$x$ is 0 and $y$ is 2.

ALICE

Exactly. So where in this plane is the value of $x$ equal to 0 and the value of $y$ equal to 2?

BOB

Oh! It's right there...

Bob points at the point $(0,2)$ in the plane.

BOB

...because that point is $(0,2)$.

ALICE

Yes. And notice that that point lies on the graph.

BOB

OK.

ALICE

So the fact that the point lies on the graph is what is communicating that $f(0) = 2$.

BOB

Oooh.

ALICE
So now let me see if you really understand. With this graph, can you tell me what $f(-1)$ is?

BOB
Still 2.

ALICE
OK, but what is it about the graph that is communicating that fact?

BOB
It's this point.

Bob points at $(-1, 2)$.

ALICE
Yes! And what are the coordinates of that point?

BOB
$(-1, 2)$.

ALICE
Now please observe, both the points $(0, 2)$ and $(-1, 2)$ lie on the graph, right?

BOB
Sure.

ALICE
But another way we can write these points is $(0, f(0))$ and $(-1, f(-1))$, right?

BOB
Sure. But who cares. $f$ of anything is always 2.

ALICE
That's true, and do you see how on this graph, this horizontal line, it doesn't matter what value of $x$ you choose. The point on the graph with that value of $x$ as its $x$-coordinate has a $y$-value of 2.

BOB
Wait what?

NARRATOR
Alice realizes that her last sentence was pretty complicated and that she said standing in place with a rather flat affect. She realizes that she could probably demonstrate the idea better with some relevant body movements and markings on the drawing.

ALICE
No matter what value of $x$ you choose...

Alice uses waves her hand left and right along the $x$-axis eventually stopping pointing at a more-or-less random place on the $x$-axis.

ALICE
...the corresponding point on the graph...

Alice lifts her finger two units toward the ceiling and points at the point on the graph.

ALICE
...has $y$-coordinate 2.

Alice slides her hand over to the point on the $y$-axis labeled 2.

BOB
Oh, yeah. I totally get it. That's why it's a horizontal line. Because $y$ never changes.

NARRATOR
Some students respond well to such body movements.

BOB
But what does this have to do with the probl...

ALICE
Shut up. I'm getting to that!

NARRATOR
It's ok for a tutor to expect some patience from students. It's also ok for a student to expect patience from a good tutor. Let's try that again Alice:

Rewind time:

BOB
But what does this have to do with the problem.

ALICE
Don't worry. We're getting to that. So the definition of the graph of a function is set of all points of the form $(x, f(x))$. So the $x$-coordinate records the input, and corresponding $y$-coordinate records the output.

BOB
OK.

ALICE
So let's take the function $f(x) = 2x + 1$. Can you tell me some points which lie on the graph?

BOB
How about $(1,3)$ and $(2,5)$.

ALICE
Yes. That's good. And now how about this?

Alice points to the squiggly graph that Bob drew which passed the Vertical Line Test.

ALICE
Let's say this is the graph of a function $f$. What is $f(1)$?

BOB
Uh, well, it depends on where $x = 1$ is.

ALICE
OK, so let's say $x = 1$ is right here.

Alice draws a small mark on the $x$-axis where $x = 1$ might plausibly be.

ALICE
What's $f(1)$?

BOB
Then $f(1)$ is the $y$-coordinate of that point there.

ALICE
Good! Now I'm just going to draw a line...

Alice draws the vertical line $x = 1$ so that it intersects the graph in a point. She marks this intersection point by going over it with the marker a bunch.

BOB

So it's the $y$-coordinate of that intersection.

ALICE

OK, great. Now let's look at this other graph. This fails the Vertical Line Test, right?

BOB

Yes.

ALICE

Can you draw me a vertical line which causes the Vertical Line Test to fail?

BOB

Yep.

Bob draws a vertical line which intersects the $S$-shaped curve in three places.

ALICE

Great. So the Vertical Line Test says that this cannot be the graph of a function, right?

BOB

Yep.

ALICE

But then, why does this vertical line, which intersects the curve in three places, mean that this is not the graph of a function?

BOB

Well, there's not just one $y$-value. There's three.

ALICE

Aha! That's right.

BOB

So what?

ALICE

So remember what you said a function was at the very start?

BOB

It's a rule for assigning outputs to inputs.

ALICE
Yes. And I said that was close to the right definition. But we see that the Vertical Line Test imposes one more condition.

BOB
Oh yeah, there can only be one output.

ALICE
YES! So what then is the definition of a function?

BOB
It's a rule which assigns a unique output to an input.

ALICE
On the nose.

BOB
That's it? I've been saying that the whole time!

ALICE
No. You really haven't.

NARRATOR
It's good to be firm with the ungrateful. Let's face it: Bob should be kissing Alice's feet.

BOB
You could have just told me the right definition from the start.

ALICE
Sure, but you learned more this way and I get paid by the hour.

BOB
Whatever. OK, so then the answer to the homework problem is easy.

Bob points back at the table:

| $x$ | $y$ |
|---|---|
| 1 | 3 |
| 2 | 3 |
| 3 | 4 |
| 3 | −4 |
| 4 | 3 |

BOB
The column marked $x$ represents the inputs and the column marked $y$ represents the outputs by contention.

ALICE
Convention.

BOB
By congestion. Whatever. Anyway, we see in our table there are two different $y$-values for the same $x$-value which would suggest two outputs for one input. So this is not a function.

ALICE
Nice.

BOB
I get it.

ALICE
Nice.

BOB
Nice.

NARRATOR
Nice.

THE END

# Lessons to Learn from "An Easily Imagined Encounter..."

There are a number of moments in *An Easily Imagined Encounter Between a Tutor and Student* which are examples of good and bad pedagogical practice. Some of the principles were noted by the narrator, and some were not. Here is a non-exhaustive list of possible pedagogical principles we might take to represent good practice.

1. Be attentive.
2. Guide students to solutions.
3. Be polite.
4. Ask questions. Don't just give answers.
5. Go general to ascertain where a student's misunderstanding of an idea lies.
6. Work with and build from existing understanding.
7. Watch for cues from body language.
8. Encourage students to make connections.
9. Ruthlessly squash fundamental misunderstandings.
10. But don't be overly pedantic. Find the right balance between being correct and being a jerk.
11. Analogies to familiar objects or ideas can be illuminating.
12. It's ok to stop and think.
13. Get students actively involved.
14. If a student isn't understanding an idea, try illustrating it with the simplest possible example.
15. Don't let frustration (which will inevitably come) get the best of you.
16. If you don't know something well enough, be open and honest in admitting it.
17. Body movements enhance the exposition of an idea.
18. Be patient.

∗ **Exercise 1:** Rank yourself on your proficiency in the practice of each of these principles based on your self-perception and, perhaps, your past experience with tutoring or teaching in general. Fill in the following table with a rating of yourself on a scale from 1 to 5, 5 representing mastery and 1 representing incompetence.

| # | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| proficiency | | | | | | | | | | | | | | | | | | |

Perhaps if you see yourself as strong in the use of one of these principles, you can use it more often. And perhaps if you see yourself as weak in one, you can try to be conscientious in how you might improve.

∗ **Exercise 2:** Are there other principles of good teaching practice you believe were not listed above, or even present in *An Easily Imagined Encounter...*?

Author: Taiyo Inoue

## 2.2 Different Approaches

**[A.K.A. Two roads diverged in a yellow wood ....]**[1]

In tutoring, there is no single way to go about guiding students towards a better understanding. In fact, there are many ways and sometimes you should try several with a student. While there may be pros and cons to some approaches, they may all be valuable. In groups of 2 or 3, consider the following storyline (think of it as a "Choose Your Own Adventure") and critique the different scenarios and what tutoring practices the tutor is using (see the worksheet on the previous page). We will discuss this as a group after you take some time to analyze what is going on.

### Exponent Rule Example

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## Beginning Question

Student: *I learned how exponents work in high school, but I'm still not getting these homework questions right...can you help me?*

### Tutor Response Option 1:

Tutor: Okay, let me just write a few examples up to talk us through this concept. (Go to **Product Rule Discussion**)

### Tutor Response Option 2:

Tutor: Can you show me a specific problem?

Student: Sure, let's do this one: What is the power of $z$ in this expression? $(x^3y^1z^7)^1(x^5y^4z^0)^7$

Tutor: Okay, why don't you tell me what your line of thinking is going into this problem.

#### Student Response 2a:

Student: I have no idea. (Go to **Product Rule Discussion**)

#### Student Response 2b:

Student: Well, I'm pretty sure when you have an exponent outside of parentheses, you get to add that exponent to all the exponents inside the parentheses.

Tutor: Let's see if we believe that hypothesis. (Go to **Product Rule Discussion**)

[1] If you don't know what we're referencing here, go check out some Robert Frost before continuing.

## Product Rule Discussion

Tutor: Let's first look at $2^3 2^2$. How can we simplify this?

Student: I'm not sure tbh.

Tutor: Okay, well what does 2 cubed really mean?

Student: It means we have three 2's.

Tutor: Okay, so let's write $2 \cdot 2 \cdot 2$. And what does 2 squared mean?

Student: Two times two.

Tutor: Okay, so we have those first three 2s, multiplied by two more, so $2{\cdot}2{\cdot}2{\cdot}2{\cdot}2$. How many 2s total?

Student: Five.

Tutor: Okay, so how do we write our answer?

Student: $2^5$.

Tutor: Great! So now looking back at the original problem, how can we get 5 from the exponents 2 and 3?

Student: You can add them!

Tutor: Exactly! In fact, this always works - anytime you see the same base (2 in our example) multiplied, you can add the exponents. This is a good thing to have jotted down in your notes, along with a few examples like these.

Tutor: Now let's look at another exponent example. (Go to **Power Rule Discussion**)

## Power Rule Discussion

Tutor: Okay, now let's consider $(2^1)^3$. How would you solve this?

Student: Well, like I said I think it would be $2^4$.

Tutor: Okay, what's another way to solve it, maybe going step by step?

Student: Hmm, well $2^3 = 8$, so then we have $(8)^1$ which is just 8.

Tutor: Great, you followed order of operations there! Now what does $2^4$ equal?

Student: Uh, well, my calculator here says 16.

Tutor: Okay, so we have two different answers here. Which method of solving do we believe?

Student: Well, I guess the one where we get the answer 8.

Tutor: I agree with you; it's easier to believe something when we can work out the steps and see each move. So I think we just need to observe what's happening with the exponents here so we can still have a rule to depend on in the case of exponents on the outside of a set of parentheses. What do you think?

Student: Well, the exponents aren't being added but I don't know what else it could be here.

Tutor: Well, looking back at our little example, what was our answer?

Student: 8

Tutor: Okay, and 2 raised to what power is 8?

Student: 3...

Tutor: Okay, and looking back at the original problem, how can we get 3 from the exponents we see there?

Student: Well, there's a 3 AND a 1....so...multiply?

Tutor: Yes, that's what I'd say too. Let's check it on one more example just to solidify this theory. Say... $(2^3)^2$.

Student: Okay, well $2^3 = 8$, and then $(8)^2 = 64$.

Tutor: Good! So, does it work to use the idea that exponents MULTIPLY in this case?

Student: Well if we add exponents, we'd get $2^5$, which does equal 64! So ya, it works!

Tutor: Awesome! I'd say that you can depend on that rule now. How about you go ahead and apply that to the problem you brought in and I'll be right here as you work through it.

Student: Well, there's so many more bases in this problem.

Tutor: ya, that can be overwhelming. But just remember, we can always depend on order of operations. So, what's the first thing we go to using order of operations?

Student: Parentheses?

Tutor: Right, okay, so in the first set of parentheses, is there any simplification we can do?

Student, Well, I guess not, since there's an $x, y$, and $z$ but not repeats of those.

Tutor: Okay, and the same for the second set of parentheses. So, now we move outside of the parentheses. What can you do there?

Student: Well, we just learned that rule that exponents multiply but there's a lot more here.

Tutor: Sure, so let me show you a trick. We can rewrite the first set of parentheses as $(x^3y^1z^7)^1 = (x^3)^1(y^1)^1(z^7)^1$.

Student: Okay, I can work with that. We get $x^3y^1z^7$.

Tutor: Great! Can you do the same with the second set of parentheses?

Student: Okay, so we rewrite $(x^5y^4z^0)^7 = (x^5)^7(y^4)^7(z^0)^7$, so then we get $x^{35}y^{28}z^0$. Wait...isn't $z^0$ just 1?

Tutor: Right! So what do we have overall?

Student: $x^3y^1z^7x^{35}y^{28}$

Tutor: Can we simplify that more?

Student: Hmm well now there ARE two $x$'s and two $y$'s...

Tutor: Right...

Student: So, from before, we can just...add the exponents to get $x^{38}y^{29}z^7$.

Tutor: Great! What questions do you have about what we just talked about?

## 2.3 Worksheet Assignment

As a last activity, we are going to ask you to fill out the following worksheet (see next page) as homework tonight. This short assignment asks you to devise strategies and reflect upon how you would respond to students with some common misunderstandings. We will talk as a group tomorrow about solutions to these types of problems.

**Example Tutoring Problem: Exponential vs. Polynomial Functions**

One problem you might encounter with students is a difficulty grasping the difference between exponential functions and polynomials.
For instance, consider the functions:

$$f(x) = x^2$$
$$g(x) = 2^x$$

In MTH 100, a student might be asked to draw $g(x)$ and draw a parabola like that of $f(x)$. Another student in MTH 210 (Calculus 1), might be asked to differentiate $g(x)$ and use the power rule.

In both cases, students are seeing a pattern in that we have a base and an exponent but they are not understanding that it matters where our variable is placed within the expression. Please answer the following questions. You may choose to consider how you would answer for a MTH 100 or MTH 210 student or both.

1. What are your first thoughts about how you might help a student understand that fundamental difference?

2. What are some leading questions you could ask them?

[Continued on next page!]

(3) What are some <u>tasks</u> you can ask them to do to guide them towards understanding?

(4) How would you try to see whether they have internalized this understanding, before moving on?

(5) Often a student who asks, “Why are we learning this at all? Why does it matter?” How would you respond to this?

# CHAPTER 3

# Concept Inventory for Math Courses

The last part of this packet is simply an inventory of concepts that cover each course we teach and some additional resources you might find useful if you are ever stumped or need to brush up on some concepts. If you are tutoring students enrolled in a particular course, we highly suggest that you::

1. Read through the concept inventory for that course and verify that you are familiar with these concepts or not. If not, it is advisable to briefly brush up on this material.
2. Use this inventory as a guide to help students in their study habits and what they need to know for their course and exams. These inventories can be thought of as a sort of "Table of Contents" or "Study Guide" for each course. *Disclaimer: these inventories contain most of the course concepts, but content and emphasis does vary slightly depending on instructor.*
3. Request the syllabus from the instructor and ask for any additional resources.

**Remember:** as a tutor, you aren't expected to remember everything in each course your students are in. You are expected to *guide* them towards understanding. So it's okay to say you don't know or you don't remember, but you should *always* give students resources or, if possible, let them know that you are going to look into it and can follow up with them later if that's useful. You are modeling the behavior of a good student: you don't give up and you seek information or clarification elsewhere if needed.

## 3.1 MTH 100

1. Functions
   a) Do you know what a function is?
   b) Given a relation or a table of inputs and outputs, can you identify which represent functions and which do not?
   c) Can you tell when a curve in the plane is the graph of a function?
   d) What is the domain of a function? Are you comfortable finding domains?
   e) What is the range of a function? Are you comfortable finding the range, particularly if you have the graph of the function in front of you?

f) Do you know how to add, subtract, multiply and divide functions?

g) Do you know how to do the composition of functions?

h) What's a piecewise function? How to work with them? How to graph them?

i) Do you know what it means for a function to be invertible?

j) Do you know how to tell if a function is invertible from its graph?

k) Do you know how to compute the inverse of an invertible function?

l) Be able to describe the relationships of inverse functions.

m) What are some examples of functions which are invertible and which are not?

n) Do you know how to compute the average rate of change of a function over some interval? Do you know how to interpret this quantity?

2. Graphs of Functions

a) Do you understand how to interpret the graph of a function?

b) If we apply some small change to $f(x)$, like adding 4 to $f(x)$ or subtracting 3 from $x$ as in $f(x-3)$, do you understand how the graph changes?

c) What about if we multiply $f(x)$ by some number like 3? What happens to the graph?

d) Know how to look at a graph and give the equation of the function you see.

e) Know how to identify $y$-intercepts and $x$-intercepts of any mathematical equation.

f) Know how to find the slope of a line.

g) Given a line, know how to find lines parallel and perpendicular.

h) Understand the relationship between shifting graphs of functions and their mathematical notation (transformation of functions).

i) Know how to identify the vertex and axis of symmetry of quadratic equations.

j) Know how to graph piecewise functions, paying careful attention to the domain restrictions.

k) Know how to determine whether a function is increasing or decreasing, and how to determine end behavior.

3. Linear Expressions

a) Do you know how to find the equation of lines given the slope and a point on the line?

b) Do you know how to find the equation of a line given two points on the line?

c) Do you know how to model phenomena using linear expressions?

d) Do you know how to solve inequalities that involve linear expressions such as $2x + 3 > 5$.

4. Quadratic Expressions

a) Do you know what a quadratic function is?

b) Do you know what the graph of a quadratic function looks like?

c) What do we mean by the roots of a quadratic?

d) When we have an equation involving a quadratic, like $x^2 + 6x + 9 = 1$, why do we strongly prefer to 0 on the right side instead of some other number?

   e) Do you know how to factor a quadratic?
   f) Do you know how to complete the square?
   g) Do you know the quadratic formula?
   h) What's an irreducible quadratic? How can we detect them?
   i) How can you use a gadget called the discriminant to tell if a quadratic has 0, 1 or 2 real roots?
   j) Know how to find the solutions/zeros of quadratic equations.

5. Equations involving radicals
   a) An equation involving radicals is something like $\sqrt{2x+3} = x$.
   b) What are "fake solutions?" Why do they arise when working with this sort of equation?
   c) Can you solve $\sqrt{2x-3} = x$?

6. Polynomial expressions
   a) Can you identify when an expression is a polynomial and when it isn't?
   b) What do we mean by the "root" or "zero" of a polynomial?
   c) What do we mean by a repeated root?
   d) What is the degree of a polynomial?
   e) Does a polynomial of degree $n$ always have $n$ real roots when taking into account repeated roots?
   f) Does a polynomial of degree $n$ always have $n$ complex roots when taking into account repeated roots?
   g) What is a good approximation for a polynomial $p(x)$ when $|x| \gg 0$.

7. Rational expressions
   a) What's a rational expression?
   b) What's a vertical asymptote? How do we find them? Does every graph of a rational expression have a vertical asymptote?
   c) What's a horizontal asymptote? How do we find them? Does every graph of a rational expression have a horizontal asymptote?
   d) After we find asymptotes, do you feel comfortable sketching the graph of a rational expression?

8. Exponential functions
   a) Can you recognize when a function is exponential?
   b) Be able to explain the components of an exponential equation.
   c) What is meant by exponential growth? What is meant by exponential decay?
   d) What are some phenomena measured by exponential growth or exponential decay?
   e) Do you know how compound interest works?

9. Logarithms

a) How are logarithms related to exponentials?
b) Know how to use logarithms to solve equations involving exponential expressions.
c) Be comfortable using the properties of logarithms in solving.
d) What's the cool thing we can say about an expression like $\log_b(a^c)$?
e) What about $\log_b(ac)$?
f) What about $\log_b(\frac{a}{c})$?

10. Trigonometry

a) Know how to describe periodic functions and their characteristics.
b) Know how to use the periodic circle to solve for various angle and side lengths of a triangle.
c) Be comfortable with the three main trigonometric functions - sine, cosine, and tangent.
d) Know how to use trigonometric properties to work with non-right triangles.
e) We have two ways of measuring angles – degrees and radians. Do you understand this?
f) Do you know how to convert an angle in degree to radians? Vice versa?
g) Do you know the definition of $\sin(\theta)$ and $\cos(\theta)$ as it relates to the unit circle?
h) Do you know the values of $\sin(\theta)$ and $\cos(\theta)$ at particularly special angles like $0°, 30°, 45°, 60°, 90°$?
i) Do you know how to use these values to find the values at other special angles in other parts of the unit circle? For example, do you know what $\sin(210°)$ is? How about $\cos(-120°)$?
j) How do we define $\tan(\theta)$? What about $\sec(\theta), \cot(\theta), \csc(\theta)$?
k) How do these trig functions relate to right triangles?
l) What is an identity?
m) The most important trig identity is the Pythagorean Identity. What does it say?
n) Do you know what the graphs $y = \sin(x)$ and $y = \cos(x)$ look like? Do you know then what the graph of $2\sin(x - \pi) + 1$ looks like?
o) Do you know how to apply trigonometry particularly to problems where right triangles are lurking?

11. Applications

a) Be able to take a contextual problem and turn it into a mathematical equation, solving as appropriate.
b) Know how to identify whether data would be better modeled linearly or exponentially.
c) Be able to translate a contextual problem into an appropriate mathematical equation and solve, interpreting your answers in context of the problem in a **clear and complete sentence.**

12. Geometry

a) Have a working understanding of distance and midpoint formulas, and Pythagorean theorem.

## 3.2 MTH 107:

1. Types of Data
   a) Do you know what the terms continuous vs. discrete mean and how they apply to data?
   b) Do you know what the terms quantitative vs. qualitative mean and how they apply to data?
   c) What is the difference between a samples and a population?
2. Measures of Central Tendency
   a) What is a mode and why is it useful?
   b) What is a median and why is it useful?
   c) What is a mean and why is it useful?
   d) What does it mean to be in the middle? How do we measure this?
   e) Is there a better or worse way to measure "the middle" of a data set?
3. Measures of Spread
   a) What is the range?
   b) What does standard deviation measure?
   c) What is the interquartile range (IQR) and what does it represent?
   d) How do we measure the spread of data? Is there one method that works better than another, or describes data better?
4. How we describe data
   a) Graphical representations: do you know how to represent different data types data graphically? Can you describe how graphical representations can be misleading?
   b) What is a distribution? What is the realtionship between distributions and their relations to graphical representations of data, measures of central tendency, and measures of spread?
   c) What does it mean for a distribution to be symmetric vs. right (positively) and left (negatively) skewed? Can we see this with measures of central tendency?
5. Measures of Error
   a) Does a sample look exactly like the population? How do we understand this?
   b) How good is the sample mean at estimating the population mean? Is the sample mean exactly equal to the population mean?
   c) How do we measure the typical error of the sample mean in estimating the population mean?
   d) What is standard error and why do we need it?
   e) Why does standard error have a $\sqrt{n}$ in the denominator? What happens to the standard error as the sample size gets large? What does this mean about the error of the sample mean in estimating the population mean?

f) What is a confidence interval?

g) What does a $z$-values or $t$-values represent? When do we use $z$ versus $t$?

6. Hypothesis testing

   a) How do we see whether our data conforms to our expectations?

   b) How do we quantify this statistically? Can we say that our statistics prove something?

   c) What are the five steps of hypothesis testing

   d) What is a null hypothesis vs. alternative hypothesis?

   e) What is a significance level?

   f) What is a critical (or "threshold") value?

   g) What does one-sided (one-tailed) vs. two-sided (two-tailed) mean?

   h) What does a $z$-values or $t$-values represent? When do we use $z$ versus $t$?

   i) What does it mean for a result to be *statistically significant*?

7. Relationships between data: correlation and regression

   a) What is correlation?

   b) What values does a correlation coefficient take?

   c) How is correlation calculated?

   d) What is causation?

   e) Give an example where correlation does not imply causation.

   f) What kind of data can we use for linear regression?

   g) Conceptually, what is a regression line? (What does it represent?)

   h) What two conditions must a line satisfy to be considered a regression line?

8. Probability

   a) What does probability mean?

   b) How do we measure probability?

   c) What is the range of a probability?

## 3.3 MTH 210:

1. Limits

   a) What do the concepts of limits and continuity mean?

   b) Can you compute limits of a variety of functions?

   c) What makes a limit "hard" to compute?

   d) What is an indeterminate form? What do we do with indeterminate forms?

   e) Can you determine regions of continuity?

   f) Can you determine end behavior using limits?

   g) Can you use l'Hopital's rule to compute a limit?

2. Derivatives
   a) Can you use derivative rules to compute the derivative of any function?
   b) Can you use a composition of derivative rules to compute more complicated derivatives?
   c) Do you know what the definition of a derivative is?
   d) Can you use the definition of derivative to find a derivative of a function?
   e) Can you determine regions of differentiability?
3. Behavior of functions and graphs
   a) Can you use the 1st derivative to sketch graphs of functions?
   b) Can you use the 2nd derivative to sketch graphs of functions?
   c) Can you determine where a function is increasing/decreasing?
   d) Can you determine where a function is concave up/dow?
   e) Can you use the 1st and 2nd derivative tests to classify extrema?
   f) Can you define critical point, inflection point, local/global extrema?
4. Applications
   a) Can you explain the relevance of derivatives in dynamic systems in the real world?
   b) Can you translate problems involving dynamic systems into problems of calculus?
   c) Can you use the derivative to solve applied problems involving dynamic systems?

## 3.4 MTH 211:

1. Integration
   a) Can you define a definite versus indefinite integral?
   b) Can you find antiderivatives using a variety of rules?
   c) Can you find areas of regions under curves and between curves?
   d) Can you approximate the area under a curve using Riemann sums?
   e) Can you find the exact area under the curve using the definition of an integral?
2. Sequences and Series
   a) Can you evaluate infinite sequences and series and be able to determine whether they converge or diverge?
   b) Can you explain what a sequence and series mean?
   c) Can you write a sequence or series using mathematical notation?
3. Applications:
   a) Can you apply definite integrals in the solution of practical problems in geometry, science and engineering?
   b) Do you understand differential equations and how we use them in mathematical modeling?

## 3.5 MTH 212:

1. You should know about vectors.
   a) What kinds of physical quantities do vectors represent?
   b) How do we notate vectors in this course?
   c) What is a unit vector? How do we find a unit vector pointing in a direction?
   d) What is the distinction between points and vectors?
   e) Do you know how to compute the dot product of two vectors? The cross product of two vectors? Does it make sense to take the cross product of vectors in $\mathbb{R}^2$? In $\mathbb{R}^3$? In $\mathbb{R}^n$?
   f) How does the dot product give rise to notions of angles and lengths?
   g) What does the dot product output? How to interpret its output?
   h) What does the cross product output? How to interpret its output?
2. You should know about lines and planes.
   a) Do you know how to find a line in the plane or space with a given direction and a given point which the line passes through?
   b) Do you know how to find a line given two points which lie on the line?
   c) Do you know how to find the equation of a plane given a normal vector and a point the plane passes through? In particular, do you remember the story that leads to the derivation of this formula?
   d) Do you know how to find the equation of a plane given three points it passes through which don't all lie on a line? Why do I need to have a condition like that?
   e) Given a plane, do you know how to find an equation of a line normal to the plane which passes through a point?
   f) Given two planes, how can you detect if they are parallel or not?
   g) If two planes intersect, do you know how to find the line of intersection?
3. You should know about parameterized curves. Particularly, vector-valued parameterizations.
   a) What are they good for?
   b) How do you parameterize a circle in the plane?
   c) How do you parameterize lines?
   d) How would you parameterize a triangle?
   e) How do you compute derivatives of parameterized curves?
   f) How do you interpret the derivative of a parameterized curve?
   g) How about velocity and acceleration? What about the normal and tangential components of acceleration?
4. You should know about scalar fields and vector fields.
   a) What are they?

b) What are some physical quantities they can represent?

c) What are partial derivatives? What do they represent? How about directional derivatives more generally?

d) What are level sets of a scalar field? What are the level sets of a the scalar field $f(x, y, z) = x + y + z$? What about $f(x, y, z) = x^2 + y^2$? What about $f(x, y, z) = x^2 + y^2 + z^2$?

e) What is the gradient of a scalar field? Why do we care about it? How do we interpret it?

f) How is the gradient of a scalar field related to the level sets?

g) How can we use the gradient to or tangent planes to surfaces?

h) Is every vector field the gradient of some scalar field?

i) How can we maximize or minimize scalar fields? On a closed and bounded region, how do you find the maximum and minimum values of a scalar field?

5. Do you feel comfortable with double and triple integrals?

a) If I give you a region $R$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ to integrate a scalar field over, can you set up the integral? Can you compute it? What does it mean?

b) Are you competent at describing regions using inequalities on coordinates?

c) Can you change order of integration of some integral?

d) When is it advantageous to use cylindrical or spherical coordinate systems?

e) What is the role that the coordinate transformation equations in this sort of problem?

f) Do you know how to find mass and center of masses of object with three dimensional shape and non-uniform density?

6. Do you like line integrals? Do you like like line integrals?

a) What is the expression for the arc length element $ds$ given a parameterization of a curve?

b) Given a scalar field $f$ and a curve $C$, do you know how to compute $\int_C f\, ds$?

c) Given a vector field $\mathbf{F}$ and an oriented curve $C$, do you know how to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$? Do you know how to interpret this integral?

d) Why is it necessary to have an orientation on the curve when integrating vector fields, but not when integrating scalar fields?

e) Do you know about the special thing that happens to line integrals when the vector field being integrated is conservative? They become very easy to evaluate.

f) What does a vector field being conservative mean anyway? Is there a nice way to check if a vector field like $\mathbf{F} = \langle P, Q \rangle$ is conservative?

g) How do you find the scalar potential of a conservative vector field?

h) What is the importance of the domain of $\mathbf{F}$ being simply-connected? What are examples of simply-connected and non-simply connected regions in the plane?

i) What is the curl($\mathbf{F}$)? How do we calculate it and how do we interpret it?

j) If the curl of a vector field is 0, is it conservative?

k) If a vector field is conservative, does it have zero curl?

l) Why is the curl of a vector field important for line integrals?

m) What is Green's Theorem? Why is it awesome?

7. Do you like surface integrals?

a) Do you know how to parameterize a surface? What if the surface is the graph of a scalar field, like $z = f(x, y)$? What if the surface is a constant coordinate surface in some coordinate system? What if the surface arises as a surface of revolution around some axis?

b) Why do we care to parameterize surfaces?

c) How do we calculate surface area of parameterized surfaces?

d) How do we calculuate integrals of vector fields over parameterized surfaces? How do we interpret such things?

e) What does an orientation on a surface mean?

f) How can we find a normal vector to an oriented surface in $\mathbb{R}^3$ which agrees with the orientation?

g) What is a closed surface? What are examples of closed and non-closed surfaces?

h) What is an orientable surface? Is every surface orientable? No? What's an example of a non-orientable surface?

i) What is the divergence of a vector field? Why do we care?

j) State the divergence theorem.

k) What is Stokes' theorem? Why do we care?

## 3.6 MTH 215:

1. First-order Ordinary Differential Equations:

a) What is the most general form for a first order differential equation?

b) Do you know how to solve a differential equation of the form $y' = f(x)$?

c) Do you know how to solve an initial value problem of the form $y' = f(x)$, $y(a) = b$?

d) Do you know how to present an antiderivative of $f(x)$ as an integral? Do you know why we might want to do such a thing? What is the solution to $y' = \exp(-x^2)$, $y(0) = a$?

e) What is a separable equation? How do you solve them?

f) What is a slope field? How do you find the slope field for a first order DE? How do solutions to the DE relate to the slope field?

g) Do you know what a general first-order linear differential equation is? Do you know why we use the word linear to describe them? Do you know how to solve them using the method of integrating factors?

h) Do you know how to apply first-order equations to problems like free-fall with air resistance, or mixing problems, or cooling problems?

2. Second-order Linear Ordinary Differential Equations

a) What is the general form for a second-order linear ODE?

b) Do solutions to second-order linear ODEs always exist? What conditions will guarantee existence and uniqueness of solutions?

c) What is a differential operator? What is a linear differential operator?

d) What does it mean for two functions to be linearly independent?

e) What is a superposition of two functions? When and why is this concept relevant?

f) What is the general strategy for solving a second-order linear ODE?

g) Given a second-order linear ODE, what is the associated homogeneous equation?

h) How do you solve a second-order homogeneous linear ODE with constant coefficients?

i) What is the characteristic polynomial? How are the roots of the polynomial relevant?

j) Do you know how to work with complex numbers?

k) What is Euler's formula? Why is it relevant to this course?

l) How do second-order linear ODE relate to harmonic oscillators?

m) What is the behavior of a free and undamped oscillator? What is the amplitude of the oscillation? What is the angular frequency of the oscillation? What is $\omega_0$ and why do we call it the natural or resonant frequency of the harmonic oscillator?

n) When is a free oscillator overdamped? Critically damped? Underdamped? For which of these cases does the oscillator oscillate? What is the frequency of oscillation if it does oscillate?

o) How do you use the method of undetermined coefficients to find a particular solution to a non-homogeneous second order linear differential equation with constant coefficients?

p) Sometimes your first guess using the method of undetermined coefficients fails. Why does this happen and what do you do in this case?

q) What is the behavior of a periodically forced undamped harmonic oscillator when the period of the forcing is different from the natural frequency? How can we see the phenomenon of beats arising in the mathematical solution in this case?

r) What is resonance? When does it happen? What can we say about the amplitude of oscillation in this case?

3. Matrices and linear systems of equations

   a) Do you understand the arithmetic of matrices? How to add? How to multiply?

   b) How do you solve the equation $Ax = b$ using elementary row operations?

   c) How do you compute the inverse of a matrix if it exists?

4. Linear Algebra

   a) Do you know what is meant by a matrix $A$ with dimensions $m \times n$?

   b) Do you know how to refer to the entries of such a matrix? What would $a_{12}$ refer to?

   c) Do you know how to do arithmetic operations with matrices? If $A$ and $B$ are matrices, when are $A + B$, or $kA$, or $AB$ well-defined? When they are well-defined, how do you calculate these?

d) Do you understand that matrix multiplication is not commutative? Do you have an example of a pair of square matrices $A$ and $B$ where $AB \neq BA$?

e) Do you know what the identity matrix is? What is its important property?

f) Do you know what the inverse of a matrix is? Does a matrix always have an inverse? If not, what is a way to detect when a matrix has an inverse?

g) Do you know how to compute the inverse of a matrix?

h) Do you know how to solve the equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x}$ and $\mathbf{b}$ are column vectors?

i) Under what conditions is there exactly one solution to $A\mathbf{x} = \mathbf{b}$?

j) Do you know what the elementary row operations are?

k) Do you know what the reduced-row echelon form of a matrix means?

l) Do you know what the determinant of a square matrix is? Do you know how to compute it (cofactor expansion)?

m) Do you know what eigenvectors and eigenvalues of a matrix are? Do you know how to find them?

n) Do you know how to work with complex eigenvalues?

o) Do you understand the issues that may arise with repeated eigenvalues?

p) Do you know what is meant by the algebraic and geometric multiplicity of an eigenvalue?

q) Do you know what a generalized eigenvector is?

r) Do you know what the exponential matrix is? Do you know how to compute it if you have an $n \times n$ square matrix with $n$ linearly independent eigenvectors?

5. Systems of First Order Linear Differential Equations

a) Do you know how to express the general form of these systems?

b) What does it mean for such a system to have constant coefficients?

c) What does it mean for such a system to be homogeneous?

d) Do you understand the superposition principle for homogenous linear systems?

e) Do you understand the notion of linear independence of solutions?

f) Do you understand how to solve $\mathbf{x}' = A\mathbf{x}$ where $A$ is an $n \times n$ matrix by analyzing the eigenvalues and eigenvectors of $A$?

g) Do you know how to incorporate initial conditions for such an equation?

h) Do you know how to transform a complex-valued solution into a pair of real-valued solutions?

i) Do you know what to do if the eigenvalues are repeated?

j) Do you know how to solve $\mathbf{x}' = A\mathbf{x}$ using the matrix exponential?

k) Do you know how to solve $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ using the method of integrating factors?

6. Fourier Series

a) What does it mean for a function $f(t)$ to be $2L$-periodic?

b) What is the period of a function like $\sin(\omega t)$?

c) Do you know what it means to extend a function defined on $[-L, L]$ to a $2L$-periodic function?
d) What is the general form of a Fourier series?
e) What functions can be represented by Fourier series?
f) What do we mean when we say two functions are orthogonal?
g) Why is orthogonality relevant?
h) What are the formulas for Fourier coefficients?
i) How can we use symmetries in $f(t)$ to simplify our calculations of Fourier coefficients?
j) Do you know how to use the Fourier series of $f(t)$ to compute solutions to solve periodically forced harmonic oscillators? In particular, do know how the linearity of the harmonic oscillator plays a role?
k) Do you know how resonance can come into the picture? Do you know how to handle it mathematically?

7. Laplace transform

a) What's the definition of the Laplace transform of $f(t)$?
b) What's the Laplace transform of 1? Of $t$? Of $e^{-at}$?
c) What's the Laplace transform of $x'(t)$? Of $x''(t)$?
d) What does the linearity of the Laplace transform mean?
e) What's the frequency shifting property of the Laplace transform?
f) What's the inverse Laplace transform? Is it linear?
g) Are you comfortable taking Laplace transforms? Are you comfortable taking inverse Laplace transforms?
h) Do you know how to find the inverse Laplace transform of a rational expression of $s$? In particular, how partial fraction decomposition and completing the square come into play?
i) Do you know how to solve differential equations with initial conditions using the Laplace transform? How can you use the solution to this sort of problem to find the general solution to the differenital equation?
j) Do you know what the unit step function is?
k) Do you know how to express piecewise defined functions using the unit step function?
l) Do you know what the Laplace transform of the unit step function is?
m) Do you know the Second Shifting property?
n) Do you know how to solve harmonic oscillators with piecewise defined forcing functions, such as the ramping force or the box force?
o) Do you know what convolution is?
p) Do you understand the meaning of the statement, "Multiplication in the frequency domain corresponds to convolution in the time domain."
q) Do you know how to use convolution to compute inverse Laplace transforms of products?
r) Do you know how to use convolution to express solutions to harmonic oscillators with a forcing function $f(t)$? Do you understand the power of this?

## 3.7 Other Course Resources

These lists can easily become out-of-date quickly, but new tools and resources are being developed all the time. A few notes:

- Videos like Khan Academy content are definitely great tools to help clarify and explain a topic to students. You can also use them to re-learn material you may be rusty on. If students use a video to try to understand the concept, definitely follow up with asking a student to do a similar example or ask them to do an example concurrently [stop the video periodically to check the work]. Don't spend too long watching videos though! The most important step in learning math is for students to DO the math themselves.
- We caution against relying on online math solutions as a way of helping students understand *how* to solve problems. You are a far better resource than something like Chegg and your goal is to model what a good student does and encourage students to be independent learners that think critically about the problem first.
- Online solutions (or back of the book solutions, for that matter) should only be checked as a last-ditch effort or if you want to make sure an answer is correct. Ultimately, students will have to be able to figure out problems without these resources (on exams, for instance) and it is important to encourage this independence at all times. Remember: on tests, students likely will have no answer key to check their work ... encourage students to think about ways they can check their work themselves without an answer key.
- Many of our courses cover content that can be found in free, online texts (for instance, Apex Calculus is a free text that covers the basics of calculus though students often need more practice than its problem sets provide). The instructor of the course may be using a free, online text, or may have some they could suggest you look at. Check out their syllabus first to verify what textbook(s) they list.
- The instructor of the course is typically more than happy to clarify anything for you as a tutor, as well. Don't be afraid to drop by their office hours or ask for an appointment.

# Appendices

# APPENDIX A

# Solutions or Discussion Ideas for the Homework Assignment

**(for the worksheet "Example Tutoring Problem: Exponential vs. Polynomial Functions"):**

1. Leading questions/tasks you can ask them (slightly change depending on the level of student):
   - ✔ Choose several values of $x$ and plug it into $g(x)$. Make sure to ask them to us both positive and negative $x$ values, if they do not do this immediately. Do your values match your graph or derivate expression? Why? What *should* the graph or derivative's functional form be, roughly?
   - ✔ Can you describe what the role of 2 is in the function? What happens as you change $x$? What happens when $x$ becomes negative?
   - ✔ Can you write this function in another way? (Ex: $f(x) = x \cdot x$, whereas $g(x) = 2 \cdot 2 \cdots$ ($x$ multiplications) ... you may need to focus on specific integer values of $x$ for simplicity).
   - ✔ Ask whether they can determine what happens as the $x$ gets more and more negative.
   - ✔ Ask whether they can determine what happens as the $x$ gets more and more positive.
2. Solidify their understanding by going through several other examples of similar functions "pairs" and why they are fundamentally so different. Ask them what is constant and what is a variable in the expressions, plug in values for $x$ and ask the student to either sketch the graph and/or describe what the derivative must do (sketching this might help).
   a) $f(x) = x^3$ vs. $g(x) = 3^x$
   b) $f(x) = x^0$ vs. $g(x) = 0^x$
   c) $f(x) = x^1$ vs $g(x) = 1^x$
   d) (*Handle with care if you choose this example*) $f(x) = x^{-1}$ vs. $g(x) = (-1)^x$. This is a hard one ... only use integer values of $x$ to avoid confusion, at least initially. If you get a really interested student, you can tell them to think rational values of $x$ but leave that for them to ponder outside of your session. It's not your goal today.
3. Test their internalization of the concept by giving some examples of mixed expressions of exponentials and polynomials and ask them to simplify the expressions (combine like terms).
   a) Simplify $2^x - 3x^2 + 3^x + 6x^2 + 3 \cdot 2^x$.

b) Simplify $2^x - 3x^2 + 3^x + 6x^2 + 4^x + x$. (More challenging)

c) Simplify $2^x - 3x^2 + 3^x + 6x^2 - 8^x + x$. (Even more challenging)

d) Simplify $2^x - 3x^2 + 3^x + 6x^2 + 10^x + x^2$. (More than one correct answer here for dealing with the $10^x$ .)

e) Simplify $2^x - 3x^2 + 3^x + 6x^2 - 6^x + 2x^2 + 9^x$. (More than one correct answer here for dealing with the $6^x$ .)

4. Return to a similar task and see how they do: ask them now to draw graphs or find derivatives of $f(x) = x^4$ and $g(x) = 4^x$.

5. Why do we care about exponential functions versus polynomials? Students may want to know or need more context for motivation. Here are some thoughts.

   a) *Applications perspective:* we use these functions for different purposes, generally speaking. Exponential functions model things like growth and decay, but are also useful in modeling oscillartory behavior such as seasonal effects or signals (electrical circuits)... though to describe why, we need to learn some more math. Exponential functions are used in business, economics, physics, biology, ... you name it ... to model and describe a wide variety of behavior. Polynomials are used in all fields as well. but typically as a simplified modeling framework or approximation. Polynomials are often used to approximate data over a limited range or to model interactions between, say, people or chemicals.

   b) *Mathematical perspective:* two very different functions that we have in our world. Polynomials are extensions of the concept of a line. Exponentials are a completely different beast. They have different behaviors, and different ranges. In math, we study the abstract science of number, quantity, and space, and the ramifications of combining basic mathematical concepts in different ways. It turns out that they way we combine concepts matters! This is what makes math interesting! It turns out that there is a *beautiful* relationship between exponential functions, trig functions and polynomials, but that is a much longer story usually covered in Calc 2 and requires investigations of $\infty$.

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